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Motion of a bore over a sloping beach: an approximate analytical approach

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The motion of a bore over a sloping beach, earlier considered numerically by Keller, Levine & Whitham (1960), is studied by an approximate analytic technique. This technique is an extension of Whitham's (1958) approach for the propagation of shocks into a non-uniform medium. It gives the entire flow behind the bore and is shown to be equivalent to the theory of modulated simple waves of Varley, Ventakaraman & Cumberbatch (1971).

1. Introduction

Whitham (1958) gave a simple rule governing the propagation of a shock wave into a non-uniform medium when the shock has assumed an essentially self-propagating character and changes mainly owing to the non-uniformity of the ambient medium. According to this rule, the compatibility condition, holding strictly along a positive characteristic, for example, is assumed to hold along the trajectory of a forward-propagating shock. Several authors (Meyer 1963; Hayes 1968; Ardavan Rhad 1970; Yousaf 1974) have attempted to explain the phenomenal accuracy of this rule when applied to specific problems. These investigations have resulted mainly in a refinement of the coefficients of the differential relation giving the shock path. Little progress by way of analytic treatment has been made beyond what was given by Whitham himself in his original paper.

In the present paper we adopt a somewhat different approach and extend Whitham's technique such that, besides the shock path, the entire details of the flow behind the shock may be obtained. We study a specific problem, namely the propagation of a bore up a sloping beach. This problem was solved numerically by Keller *et al.* (1960). Our method derives from Whitham's explanation for his rule. If $u(x,t)$ and $g^{-1}c^2(x, t)$ denote the fluid velocity and disturbed depth of water respectively, Whitham showed that his rule gave good results because the factor $(u+2c)_{,t}$ which appears in the difference between the compatibility condition along the positive characteristic and the equation along the bore obtained by his rule is small all along the bore. We assume that this factor is small everywhere behind the bore and obtain the interesting result (§2) that this approximation is precisely that of Varley *et al.* (1971) and gives their modulated simple wave. Varley *et al.* employed their theory of modulated simple waves to study the propagation of a large amplitude, high frequency, boreless pulse up a sloping beach. We

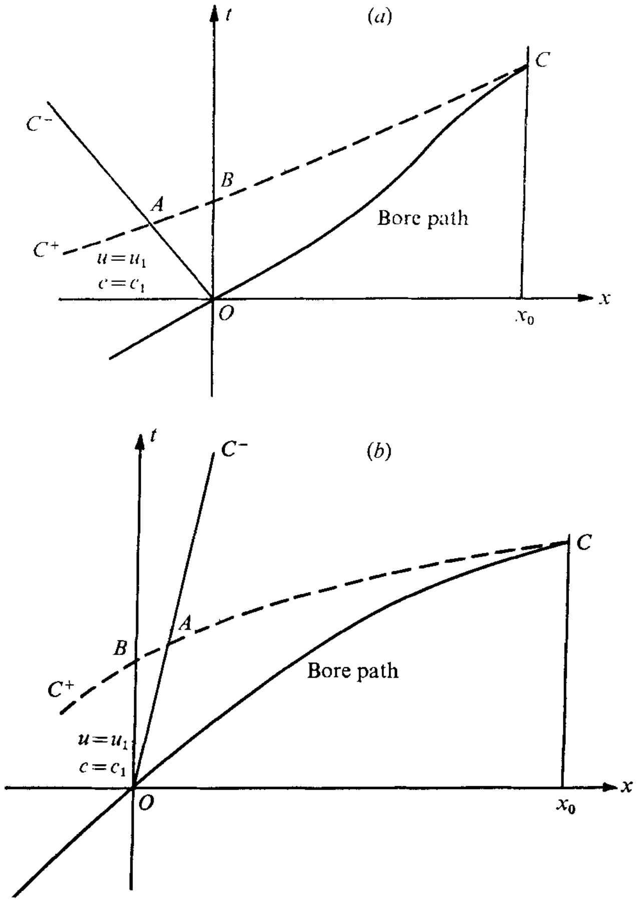

FIGURE 1. The x, t diagram for (a) $u_1 < c_1$ and (b) $u_1 > c_1$.

show that the same approximation gives excellent results for flows headed by a bore, is consistent with Whitham's shock rule and satisfies the boundary conditions exactly at the interface separating the non-uniform disturbed flow behind the bore and the uniform flow left behind by an initially uniform bore. Our numerical results, given in §4, show good agreement with the exact numerical solution of Keller *et al.*

2. Theory

Consider the propagation of a bore which is uniform until it reaches $x = 0$, after which it encounters a sloping beach with slope $h_0'(x)$ for $0 < x \leqslant x_0$. If $g^{-1}c^2$ is the disturbed depth of water and u is the fluid velocity, the equations of shallow-water theory are

$$u_{,t} + (c^2 + \tfrac{1}{2}u^2)_{,x} = gh_0'(x), \tag{2.1}$$

$$c^2_{,t} + (uc^2)_{,x} = 0. \tag{2.2}$$

The conditions at the bore are

$$uc^2 = U(c^2 - gh_0), \tag{2.3}$$

$$u^2c^2+\tfrac{1}{2}c^4-\tfrac{1}{2}(gh_0)^2 = U(uc^2), \tag{2.4}$$

where U is the bore velocity. The propagation problem referred to above may be stated mathematically as a boundary-value problem for the system (2.1) and (2.2), with boundary conditions (2.3) and (2.4) at the bore and uniform conditions $u = u_1$ and $c = c_1$ at the C^- characteristic OA for the case when the bore is initially subcritical (see figure 1*a*). The corresponding boundary-value problem for the case when the bore is initially supercritical is shown in figure 1 (*b*).

We substitute the Riemann variables $F = u+2c$ and $S = u-2c$ in (2.1) and (2.2), and choose the negative-characteristic co-ordinate α and the spatial co-ordinate x as the new independent variables. The co-ordinate α is parametrized such that $\alpha = t$ at the bore. The system (2.1) and (2.2) now becomes

$$F_{,\alpha} = \frac{(3S+F)\,[\tfrac{1}{4}(3F+S)\,F_{,x} - gh_0'(x)]}{2(F-S)\,\alpha_{,t}}, \tag{2.5}$$

$$\tfrac{1}{4}(3S+F)\,S_{,x} = gh_0'(x). \tag{2.6}$$

Equation (2.5) may be rewritten as

$$F_{\alpha}\alpha_{,t}\left[\frac{1}{U}-\frac{4}{3S+F}\right]+F_{,x}-\frac{4gh_0'(x)}{3F+S} = \left[\frac{1}{U}-\frac{4}{3F+S}\right]F_{,\alpha}\,\alpha_{,t}. \tag{2.7}$$

Whitham's rule is obtained by equating the left-hand side of (2.7) to zero. Along the bore this rule gives

$$\frac{dF}{dx}-\left[\frac{4g}{3F+S}\right]\frac{dh_0}{dx} = 0. \tag{2.8}$$

Integration of (2.8), together with the conditions (2.3) and (2.4), gives the solution

$$F = F_B(x), \quad S = S_B(x) \tag{2.9}$$

just behind the bore, and the bore path itself is obtained as

$$t = t_B(x), \tag{2.10}$$

where the initial condition $F_B(0) = u_1+2c_1$ has been employed.

Whitham has argued that his rule works because the factor $F_{,\alpha}\ \alpha_{,t} = F_t$ is small all along the bore. To connect his approximation with the theory of modulated simple waves of Varley *et al.* we rewrite (2.6) as

$$[1+\Delta F/(3S+F_B)]\,(3S+F_B)\,S_{,x} = gh_0'(x), \tag{2.11}$$

where $\Delta F = F-F_B$. Now we assume that

$$|\Delta F/(3S+F_B)| \ll 1, \tag{2.12}$$

defining a modulated simple wave (Varley *et al.* 1971). While Whitham's rule is based on $F_{,t}$ being small for each point x on the bore path, the assumption (2.12) further requires ΔF to be small over the time interval of interest. Thus the flow in the entire region behind the bore is a modulated simple wave with $F = F_B(x)$, and is easily found. The results obtained under this assumption agree very well with exact numerical solutions (§4).

Under the assumption (2.12), (2.11) may be approximated by

$$(3S+F_B)\,S_{,x} \doteqdot gh_0'(x). \tag{2.13}$$

Starting from a point (x, α) on the bore, (2.13) is integrated along the negative characteristic α = constant. The initial condition at the bore is

$$S(x,\alpha)_{\text{bore}} = S_B(x) \tag{2.14}$$

and is obtained from (2.9). To a first approximation we have

$$t(x,\alpha) = \alpha + 4\int_{x_B}^{x} \frac{dx}{(F_B+3S)}, \tag{2.15}$$

where $\alpha = t_B(x_B)$; see (2.10).

The solution in the entire region behind the bore is obtained as follows. In the subcritical case we proceed from the bore to the line $x = 0$ as explained above (see figure 1*a*). The region *AOB* bordering the constant state on the left is a simple wave and is given by

$$\begin{gathered} x = t(\alpha, 0) + \tfrac{1}{4}(F+3S)\,t(\alpha, x), \\ F = u_1 + 2c_1, \quad S(\alpha, x) = S(\alpha, 0). \end{gathered} \tag{2.16}$$

In the supercritical case (figure 1*b*) the solution in the region *AOC* is obtained as explained above and that in the region *AOB* is steady and, therefore, given by

$$S = S(0,x), \quad F = F_B(x). \tag{2.17}$$

We notice from (2.16) and (2.17) that the boundary conditions on the lines separating uniform regions, *OA* for the subcritical and *OB* for the supercritical case, have been exactly satisfied.

The error involved in the approximation (2.12) is estimated from (2.5), making use of (2.8). Thus, at a fixed station $x = x_1 > 0$,

$$\left|\frac{\Delta F}{3S+F_B}\right| = \left|\frac{gh_0'(x)}{3S+F_B}\int_{t_B(x_1)}^{t_s(x_1)} \frac{(3S+F_B)(S-S_B)}{2(F_B-S)(3F_B+S_B)}\,dt\right|, \tag{2.18}$$

where $t_s(x_1)$ is the time at which the bore reaches the shore. Equation (2.18) implies

$$\left|\frac{\Delta F}{3S+F_B}\right| \leqslant \left|\frac{gh_0'(x)\,\tau}{3S+F_B}\right| \max\left|\frac{(3S+F_B)(S-S_B)}{2(F_B-S)(3F_B+S_B)}\right|, \tag{2.19}$$

where $\tau = t_B(x_1) - t_s(x_1)$ and the maximum is taken over $t_B \leqslant t \leqslant t_s$ at $x = x_1$. The inequality (2.19) is similar to inequality (4.21) of Varley *et al.* (1971) and shows that the approximation is good when the pulse duration is small.

In the subcritical case, the approximation (2.12) breaks down in a small region where the flow is approximately sonic (i.e. $3S+F_B \approx 0$). For example, for the case $N = 0{\cdot}25$ (see §4), this happens at the bore at $T = 0{\cdot}898$, when the bore is close to the shoreline. Keller *et al.* have depicted their numerical solution up to this time only in their figure 4(*a*). A new dependent variable $G = u - c$ is introduced (cf. Friedman 1960) and (2.6) is then written in the form

$$4G_{,x} = 3gh_0'(x)\left[\frac{1}{G} + \frac{1}{2F_B+G_B}\right] - (F_B - F)_{,x}.$$

Equations (2.5) and (2.6) are now approximated by

$$F = F_B(x)$$

and
$$4G_{,x} = 3gh_0'(x)\left[\frac{1}{G}+\frac{1}{2F_B+G_B}\right]. \tag{2.20}$$

3. Expansion near the bore

It is of some interest to compare our results with those obtained by an expansion of the solution near the bore. We assume

$$F = F_B(x) + (t-t_B(x))\,F_1(x) + (t-t_B(x))^2 F_2(x) + \ldots, \tag{3.1}$$

$$S = S_B(x) + (t-t_B(x))\,S_1(x) + (t-t_B(x))^2 S_2(x) + \ldots, \tag{3.2}$$

and substitute in the equations

$$F_{,t} + \tfrac{1}{4}(3F+S)\,F_{,x} = gh_0'(x), \tag{3.3}$$

$$S_{,t} + \tfrac{1}{4}(3S+F)\,S_{,x} = gh_0'(x), \tag{3.4}$$

which follow from (2.1) and (2.2). We obtain

$$F_1(x) + \tfrac{1}{4}(3F_B+S_B)\,(F_{B,x} - F_1\,dt_B/dx) = gh_0'(x), \tag{3.5}$$

$$S_1(x) + \tfrac{1}{4}(3S_B+F_B)\,(S_{B,x} - S_1\,dt_B/dx) = gh_0'(x). \tag{3.6}$$

It is easy to check, using (2.8), that

$$F_1(x) = 0, \tag{3.7}$$

$$S_1(x) = \frac{gh_0'(x) - \tfrac{1}{4}(3S_B+F_B)\,S_{B,x}}{1-\tfrac{1}{4}(3S_B+F_B)\,dt_B/dx}. \tag{3.8}$$

Equation (3.7) is, in fact, our assumption (2.12). The solutions (3.1) and (3.2) for both the subcritical and the supercritical case were obtained and compared with the exact numerical solution of Keller *et al.* and our approximate solution. While for the subcritical case the accuracies of this expansion and the present method are comparable, the error in each being less than 3 %, for the supercritical case the above expansion fails badly. For the first approximation the departure from the exact numerical solution becomes as large as 32 %, while the method suggested in § 2 gives a solution with an error never exceeding 4 %. Besides, $S_1(x)$ is not small compared with $S_0(x)$, so that the flow behind the bore is essentially unsteady.

4. Numerical results

In this section we give the numerical results obtained by our method for the problem discussed by Keller *et al.* A bore, initially at the point $x = 0$, climbs up a beach of constant slope β to the shoreline $x = x_0$. The flow behind the bore obtained by our method is compared with that calculated by Keller *et al.* using a

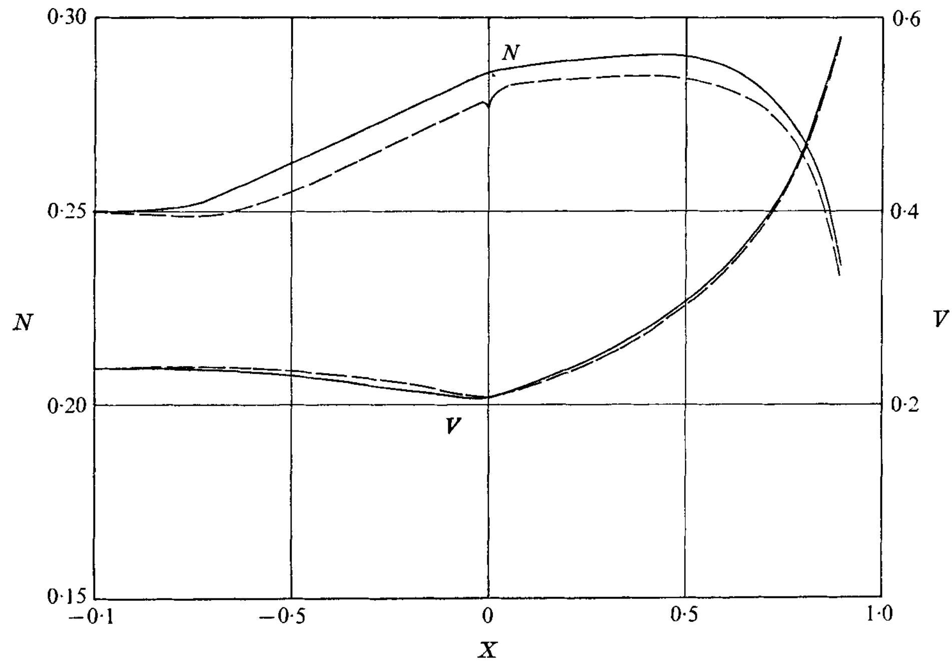

FIGURE 2 (*a*). For legend see facing page.

finite-difference scheme. To facilitate the comparison we use their notation:

$$X = x/x_0, \quad T = (t/x_0)\, g^{\frac{1}{2}} h_0^{\frac{1}{2}}(0),$$
$$v(X,T) = u/g^{\frac{1}{2}}h_0^{\frac{1}{2}}(0), \quad H(X,T) = h(x,t)/h_0(0),$$
$$V = U/g^{\frac{1}{2}}h_0^{\frac{1}{2}}(0), \quad N(X,T) = (h-h_0)/h_0(0),$$
$$H(X) = \frac{h_0(x)}{h_0(0)} = \begin{cases} 1 & (X<0), \\ 1-X & (0 \leqslant X \leqslant 1), \end{cases}$$

where $h(x,t) = g^{-1}c^2(x,t)$. Computations were carried out for two different initial heights of the bore ($N_1 = 0{\cdot}25$ and $N_2 = 10{\cdot}0$) using (2.13) and our results are compared with the exact numerical solution in figures 2 (*a*) and (*b*). The agreement is very good and the maximum deviation is less than 4 %.

5. Conclusions

We have adopted an approximate analytic approach to the problem of a bore climbing up a sloping beach, extending for the first time Whitham's (1958) characteristic rule to provide flow details in the entire region behind the bore. This approach turns out to be equivalent to the theory of modulated simple waves of Varley *et al.* (1971). The numerical results obtained by using the present approach as described in §4 show excellent agreement with the exact numerical solution of Keller *et al.* both when the initial uniform flow behind the bore is supercritical and when it is subcritical. An upper bound for the error in the

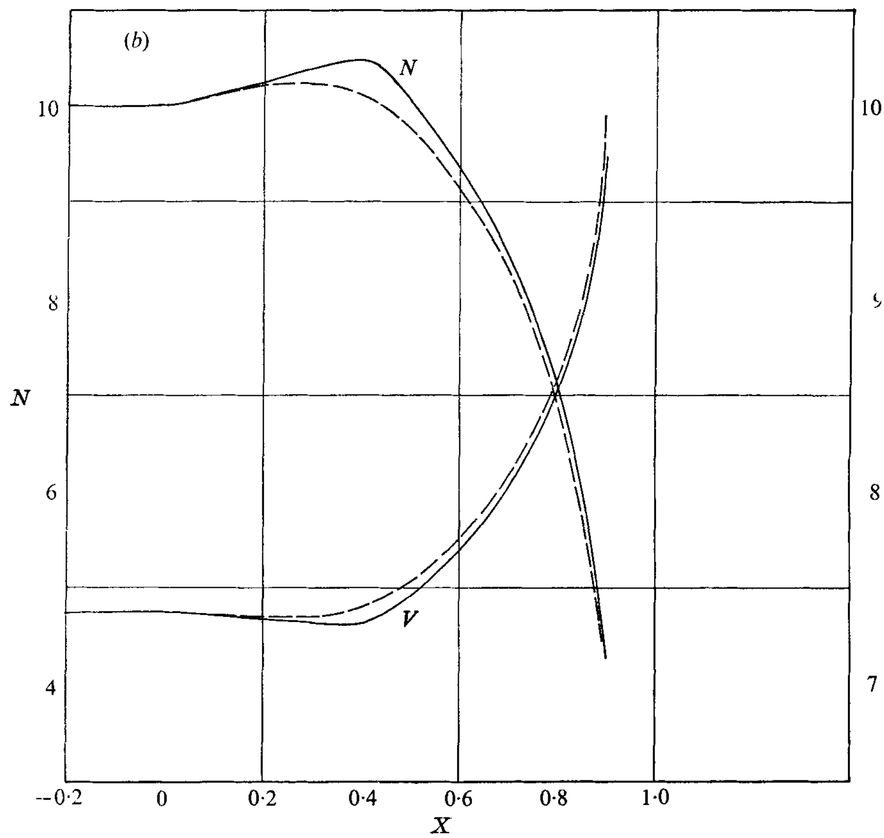

FIGURE 2. The flow behind the bore: height N and particle velocity V. – – –, Keller *et al.*; ——, present theory. (*a*) $N(0) = 0{\cdot}25$, $T = 0{\cdot}898$. (*b*) $N(0) = 10{\cdot}0$, $T = 0{\cdot}103$.

Riemann variable F based on (2.19) is found to be very small indeed. However, no attempt has been made to include the situation where secondary bores form in the sonic region (Friedman 1960).

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